WA Exams Practice Paper C, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(3 marks)**

The rate of change of temperature, T °C, of a solid placed in an oven can be modelled by $T'(t) = 2e^{0.125t}$, where t is the time in minutes since the solid was first placed in the oven.

Determine the increase in temperature of the solid during the tenth minute, rounding your answer to two decimal places.

$$\Delta T = \int_{T1}^{T2} T'(t)\ dt$$

$$= \int_{9}^{10} \left(2e^{0.125t}\right)\ dt$$

$$\approx 6.56°C$$

Question 10 **(8 marks)**

(a) State, with reasons, whether or not the following functions could represent the probability distribution of a discrete random variable X.

(i) $f(x) = \frac{x}{10},\ x \in \{\ -2,\ 0,\ 1,\ 2,\ 3,\ 6\}$. (1 mark)

No, as $P(X = -2) = -0.2$, but probabilities must be between 0 and 1.

(ii) $f(x) = \frac{1}{4},\ x \in \{\ 0,\ \frac{1}{2},\ \frac{3}{2},\ \frac{7}{2}\}$. (1 mark)

Yes, probabilities are all between 0 and 1 and sum to 1.

(iii) $f(x) = \frac{1}{2},\ 1 \le x \le 3$. (1 mark)

No, as the function is continuous between 1 and 3, rather than discrete.

(b) A student noticed that in a particular class they attended four days a week, the chance of being set homework on any one day was independent of the previous day and could be modelled by scoring a prime number when a fair ten-sided die marked with integers from 1 to 10 was rolled once.

Determine the probability that the student is set homework in this class

(i) the next time they attend. (1 mark)

$P = \frac{4}{10}$ *(Note primes are 2, 3, 5 and 7)*

(ii) exactly once in the next week. (2 marks)

$X \sim B(4,\ 0.4)$

$P(X = 1) = 0.3456$

(iii) at least five times over the next fortnight. (2 marks)

$X \sim B(8,\ 0.4)$

$P(X \ge 5) = 0.1737$

Question 11 **(9 marks)**

A transport company uses the same type of tyre for all 35 of its trailers. The number of kilometres that a new tyre lasts is normally distributed with a mean of 85 000 km and a standard deviation of 9 500 km.

(a) What percentage of all tyres used will last more than 100 000 km? (1 mark)

$$P(X > 100000) = 0.0572$$
$$\approx 5.72\%$$

(b) Two tyres are chosen at random. What is the probability that neither tyre will last for more than 100 000 km? (2 marks)

$$(1 - 0.0572)^2 = 0.9428^2$$
$$= 0.8889$$

(c) Determine the distance that will be exceeded by 99% of all tyres. (2 marks)

$$P(X > k) = 0.99$$
$$k = 62899.7$$
$$\approx 62\ 900 \text{ km}$$

(d) Given that a tyre has already travelled 90 000 km, what is the probability that it will not last another 5 000 km? (2 marks)

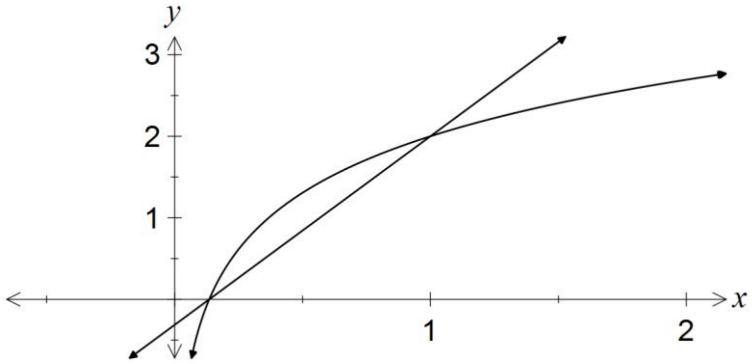
$$P(X < 95000 \mid X > 90000) = \frac{P(90000 < X < 95000)}{P(X > 90000)}$$
$$= \frac{0.15308}{0.29933}$$
$$= 0.5114$$

(e) A trailer is fitted with 12 randomly chosen new tyres. Calculate the probability that at least two of these tyres will last more than 100 000 km. (2 marks)

$$Y \sim B(12,\ 0.0572)$$
$$P(Y \geq 2) = 0.1477$$

Question 12 **(7 marks)**

The diagram below shows the graphs of $y = f(x)$ and $y = g(x)$, where $f(x) = \ln(x) + 2$ and $g(x) = ax + b$.



(a) Given that $f(x) = g(x)$ when $y = 0$ and when $y = 2$, determine the values of the constants a and b rounded to three decimal places. (4 marks)

$\ln(x) + 2 = 2 \Rightarrow x = 1$

$\ln(x) + 2 = 0 \Rightarrow x = e^{-2} \quad (\approx 0.1353)$

$$a = \frac{2}{1 - e^{-2}} \approx 2.313$$

$$y - 2 = \frac{2}{1 - e^{-2}}(x - 1) \Rightarrow b = 2 - \frac{2}{1 - e^{-2}} \approx -0.313$$

(b) Determine the area trapped between the graphs of $y = f(x)$ and $y = g(x)$.

(3 marks)

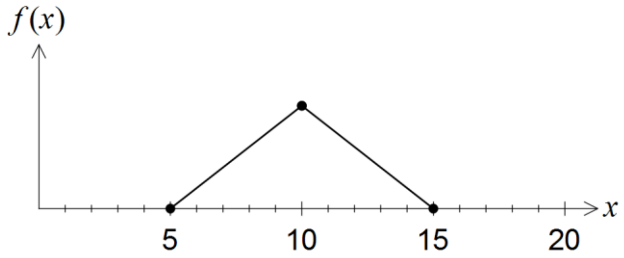
$$\int_{0.1353}^{1} \left(\ln(x) + 2 - (2.313x - 0.313)\right) dx \approx 0.271$$

Or

$$\int_{e^{-2}}^{1} \left(\ln(x) + 2 - \left(\frac{2}{1 - e^{-2}}x - \left[2 - \frac{2}{1 - e^{-2}}\right]\right)\right) dx = \frac{2}{e^{2}}$$

Question 13 **(8 marks)**

(a) A continuous random variable X is defined on the interval [5, 15] and has a symmetrical triangular probability density function as shown.



(i) Determine an expression for the probability density function of X. (3 marks)

$$f(x)=\begin{cases} \dfrac{x}{25}-\dfrac{1}{5}, & 5\le x\le 10 \\ \dfrac{3}{5}-\dfrac{x}{25}, & 10\le x\le 15 \\ 0 & \text{elsewhere} \end{cases}$$

(ii) Determine $P(X<11.5)$. (2 marks)

$$\int_{10}^{11.5}\frac{3}{5}-\frac{x}{25}\,dx$$

0.255

ans+0.5

0.755

$P(X<11.5)=0.755$

(b) Another continuous random variable W is defined by the probability density function $g(w)=\dfrac{1}{4}-\dfrac{w}{16}$, $-2\le w\le 2$. Calculate the exact mean and variance of W. (3 marks)

Define f(x)=1/4−x/16

done

$$\int_{-2}^{2} x\times f(x)\,dx$$

$-\frac{1}{3}$

$$\int_{-2}^{2} f(x)\times\left(x+\frac{1}{3}\right)^2 dx$$

$\frac{11}{9}$

Mean is $-\frac{1}{3}$ and variance is $\frac{11}{9}$.

Question 14 **(7 marks)**

According to recent research, 14% of Australians are left-handed.

(a) Explain why it is likely that the research involved some form of random sampling. (1 mark)

Determining the handedness of every Australian would be difficult, expensive and time consuming and so it is likely that random sampling would be used.

(b) If a large number of random samples of 35 Australians are collected, what proportion of these samples are expected to contain less than 10% of left-handers? (2 marks)

$$\hat{p} \sim N(p, s^2)$$
$$p = 0.14$$
$$s = \sqrt{\frac{0.14 \times 0.86}{35}} = 0.0587$$
$$P(\hat{p} < 0.10) = 0.2476$$

(c) If a large number of random samples of 300 Australians are collected, what proportion of these samples are expected to contain between 12% and 15% of left-handers? (2 marks)

$$\hat{p} \sim N(p, s^2)$$
$$p = 0.14$$
$$s = \sqrt{\frac{0.14 \times 0.86}{300}} = 0.0200$$
$$P(0.12 < \hat{p} < 0.15) = 0.5321$$

(d) One of the answers to (b) or (c) should be treated with some caution. State which answer and explain why. (2 marks)

Answer (b). The sample size n is fairly small (35), and p is not close to 0.5 ($np < 5$) and so the assumed normal distribution of sample proportions is unlikely to be good.

Question 15 **(6 marks)**

The population P, in thousands, of a city was observed to grow according to the model $P = 23.5e^{kt}$, where t is the time in months from January 1, 2004.

(a) What was the population of the city on January 1, 2004? (1 mark)

$$P(0) = 23.5e^{0}$$
$$= 23.5$$
Population was 23 500.

(b) Show that this growth model satisfies the differential equation $\frac{dP}{dt} = kP$. (1 mark)

$$\frac{dp}{dt} = k \times 23.5e^{kt}$$
$$= kP$$

During 2004, the population of the city increased by 680 people.

(c) Determine the value of k, rounding your answer to three significant figures. (2 marks)

$$23.5 + 0.680 = 23.5e^{12k}$$

$$k \approx 0.00238$$

(d) At what rate, in people per month, was the population of the city growing on January 1, 2005? (2 marks)

$$\frac{dP}{dt} = kP$$
$$= 0.00238 \times (23.5 + 0.680)$$
$$= 0.0575$$

$$= 57.7 \text{ people per month}$$

Question 16 **(8 marks)**

The clarity, C, on a scale from 0 to 10, of a sample of water containing suspended solids can be modelled by $C = 1.7\ln(t+2),\ 0 \le t \le 300$, where t is the time in minutes since the sample was left to settle.

(a) What is the initial clarity of the water? (1 mark)

$C = 1.7\ln 2 \approx 1.18$

(b) Calculate, to the nearest minute, how long it takes for the clarity of the water to first exceed 8. (2 marks)

$8 = 1.7\ln(t+2) \ \Rightarrow\ t = 108.6$

109 minutes.

(c) At what rate is the clarity of the water changing after ten minutes? (2 marks)

$$\frac{dC}{dt} = \frac{1.7}{t+2}$$

$t = 10,\ \dfrac{dC}{dt} = \dfrac{1.7}{12} \approx 0.142$ units per minute

(d) Determine the clarity of the water at the instant the rate of change of clarity first falls below one hundredth of a unit per minute. (2 marks)

$$\frac{1.7}{t+2} = \frac{1}{100} \ \Rightarrow\ t = 168$$

$$C = 1.7\ln 170 \approx 8.73$$

(e) If a clarity reading of 10 equates to perfectly clear water, explain why it is necessary to restrict the domain of the function used in the model. (1 mark)

The range of the function is infinite and so would increase past 10.

Question 17 **(8 marks)**

(a) The discrete random variable X has the probability distribution shown below.

x	1	2	3	4
$P(X = x)$	a	$2a$	b	$3b$

Determine the value of the constants a and b if $E(X) = 3$. (3 marks)

a+2a+b+3b=1

3·a+4·b=1

a+4a+3b+12b=3

5·a+15·b=3

$\begin{cases} 3\cdot a+4\cdot b=1 \\ 5\cdot a+15\cdot b=3 \end{cases}\Bigg|_{a,\,b}$

{a=0.12, b=0.16}

(b) A scratchie costing \$5 has three panels that are covered, each of which contain a picture of a diamond, heart, spade or club with probabilities 0.1, 0.2, 0.3 and 0.4 respectively. When the covering is scratched off, the pictures on the three panels are revealed. The prizes for certain combinations are shown in this table:

Combination	3 diamonds	3 hearts	3 spades	3 clubs
Prize	\$40	\$30	\$20	\$10

(i) If X is the amount gained or lost, in dollars, when buying and then claiming a prize, complete the following probability distribution table. (3 marks)

x	35	25	15	5	-5
$P(X = x)$	0.001	0.008	0.027	0.064	0.9

Complete top row with 25, 15 and 5.

$P(SSS) = 0.3^3 = 0.027$

$P(CCC) = 0.4^3 = 0.064$

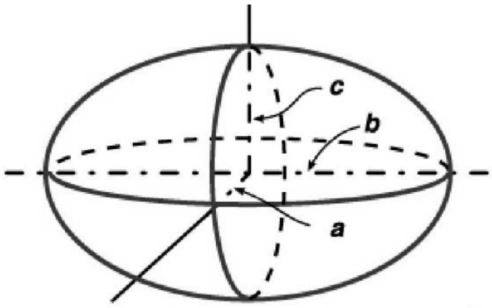
$P(X = -5) = 1 - (0.001 + 0.008 + 0.027 + 0.064) = 0.9$

(ii) Determine the expected gain or loss when purchasing a scratchie. (2 marks)

$E(X) = -3.54$, so expect a loss of \$3.54.

Question 18 **(8 marks)**

A general ellipsoid has semi-axes lengths $a,\ b$ and c, as shown in the diagram below and has volume given by $V = \dfrac{4\pi abc}{3}$.



Consider the ellipsoid where the relationship between the semi-axes lengths is that b is three times a, and that the sum of a and c is 42 cm.

(a) Show that the volume of this ellipsoid is given by $168\pi a^2 - 4\pi a^3$. (2 marks)

$$V = \frac{4\pi abc}{3}$$
$$= \frac{4\pi a(3a)(42-a)}{3}$$
$$= 168\pi a^2 - 4\pi a^3$$

(b) Use calculus to determine the length of a that maximises the volume of the ellipsoid and state the maximum volume. (3 marks)

$$\frac{dV}{da} = 336\pi a - 12\pi a^2$$

$$336\pi a - 12\pi a^2 = 0 \ \Rightarrow \ a = 0,\ a = 28$$

$V = 43904\pi \quad \approx 138000$ cm^3 (3 sf)

(c) Use the increments formula $\delta y = \frac{dy}{dx}\delta x$ to estimate the change in volume of the ellipsoid when a increases from 30 cm to 30.5 cm. (3 marks)

$a = 30, \quad \delta a = 0.5$

$$\delta V = \frac{dV}{da}\delta a$$
$$= \left(336\pi a - 12\pi a^2\right)\delta a$$

$$= (336 \times 30 - 12 \times 30^2)\pi \times 0.5$$
$$= -360\pi \quad \approx -1131 \text{ cm}^3$$

A decrease in volume of 1131 cm^3.

Question 19 **(9 marks)**

(a) A newspaper claimed that 17% of drivers never paid their parking fines. An independent body was asked to check this figure by taking a random sample and to report back with a 90% confidence interval. Determine the sample size required to ensure a margin of error of no more than 5%. (2 marks)

$$n = \frac{1.645^2 \times 0.17(1-0.17)}{0.05^2}$$

$$n = 152.7$$

Ask 153 drivers.

(b) Following a random sample of students at a variety of schools, a 95% confidence interval calculated for the proportion of CAS calculators that had been upgraded to use the most recent operating system was (0.361, 0.539).

(i) Determine the number of CAS calculators in the sample that had been upgraded. (4 marks)

$$p = \frac{0.361+0.539}{2} = 0.45$$

$$0.539 - 0.45 = 0.089$$

$$0.089 = 1.96\sqrt{\frac{0.089(1-0.089)}{n}}$$

$$n = 120$$

$0.45 \times 120 = 54$ CAS calculators.

(ii) A year prior to this sample, another survey found that 89 out of a random sample of 227 CAS calculators had the most recent operating system installed. Does evidence exist to suggest that the proportion of calculators with the most recent OS is changing? Justify your answer. (3 marks)

$$89 \div 227 = 0.392$$

$$0.392 \pm 1.96\sqrt{\frac{0.392(1-0.392)}{227}} \approx (0.33, 0.46)$$

No evidence to suggest change, as both 95% intervals have considerable overlap.

Question 20 **(8 marks)**

(a) A student simulated taking 36 random samples from a population in which it was known that one out of three people were overweight, from which a 90% confidence interval for the proportion of people who were overweight was calculated. The intervals obtained by the student after repeating the simulation ten times are shown below, rounded to two decimal places.

(0.20, 0.46)	(0.18, 0.43)	(0.26, 0.52)	(0.15, 0.40)	(0.11, 0.34)
(0.09, 0.30)	(0.28, 0.55)	(0.23, 0.49)	(0.26, 0.52)	(0.31, 0.58)

(i) Explain whether the variation in these intervals is as expected. (2 marks)

Yes.

We expect variation between intervals, and nine out of the ten of the intervals shown contain the population proportion of 0.33, which is as expected when a 90% interval is calculated.

(ii) The margin of error for one of the intervals is 0.115, but for another is 0.13. Explain why this is expected. (2 marks)

E depends on (i) the z-score (constant, 1.645) (ii) the sample size (constant, 36) and (iii) the proportion p (variable). As p varies, so the value of $p(1-p)$ varies, giving rise to different margins of error.

(b) A state university took a random sample of 162 recent PhD graduates and found that 17 of them were unemployed.

(i) Calculate a 98% confidence interval for the proportion of all recent PhD graduates from the university who are unemployed. (2 marks)

$17 \div 162 = 0.105$

$$0.105 \pm 2.326\sqrt{\frac{0.105(1-0.105)}{162}} \approx (0.049,\ 0.161)$$

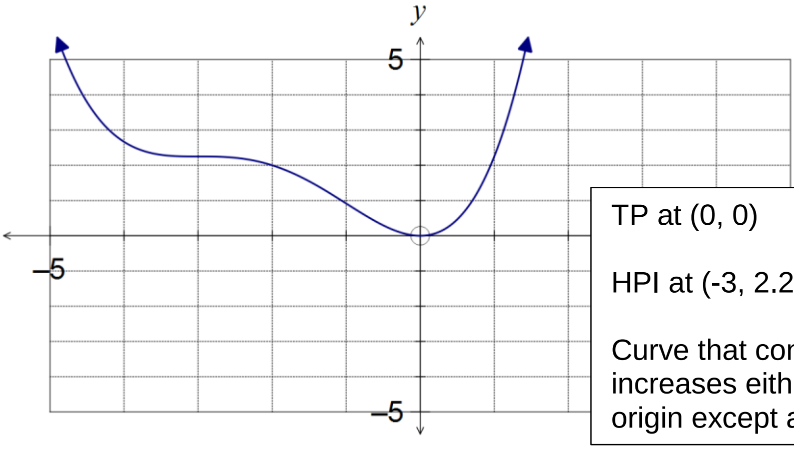
(ii) The unemployment rate is the state is 4%. Is there evidence to conclude that recent PhD graduates from the university are more likely to be unemployed than the population in general? Justify your answer. (2 marks)

Yes. The population rate falls outside the interval calculated in (i) and so it is likely the university is sampling from a different pool. We can conclude that PhD graduates are more likely to be unemployed.

Question 21 **(9 marks)**

The graph of the continuous polynomial function $y = f(x)$ has a local minimum at $(0,0)$, a horizontal point of inflection at $(-3\,,2.25)$ and no other stationary points.
The graph also has a point of inflection at $(-1\,,1)$.

(a) Sketch a possible graph of $y = f(x)$ on the axes below. (3 marks)



TP at (0, 0)

HPI at (-3, 2.25)

Curve that continuously increases either side of origin except at HPI.

(b) State the coordinates of all roots of the graph of $y = f\,'(x)$. (2 marks)

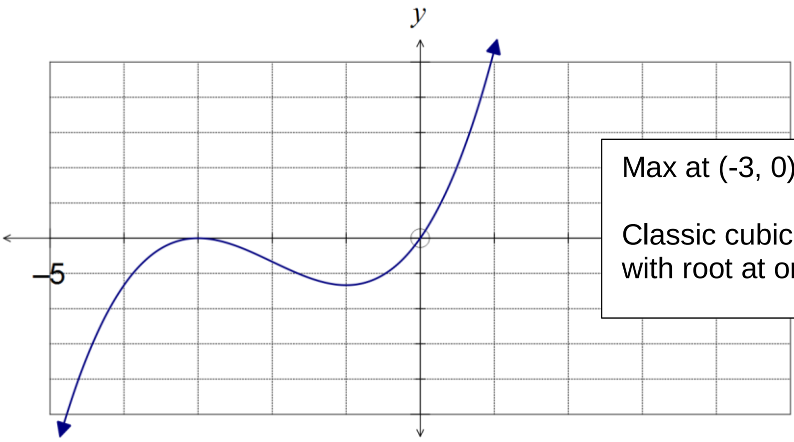
At (-3, 0) and (0, 0).

(c) Will $f\,''(0)$ be negative, zero or positive? Justify your answer. (2 marks)

Positive.

A minimum at (0, 0) indicates that curve must be concave up and so $f\,''(0) > 0$.

(d) Sketch a possible graph of $y = f\,'(x)$ on the axes below. (2 marks)



Max at (-3, 0)

Classic cubic shape with root at origin.

End of questions

Additional working space

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